Circle Toss on Grid of Squares

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This problem is adapted and expanded from the Varsity Math (Week 117) column of the 9dec17 *Wall Street Journal*. In it a circular disk is tossed randomly onto a grid of unit squares. If the disk intersects either two or three squares, then it scores. Else it does not score. In Figure 1 disks 1 and 2 score, and 3 and 4 do not score. What radius disk has the highest probability of scoring, and what is that probability?

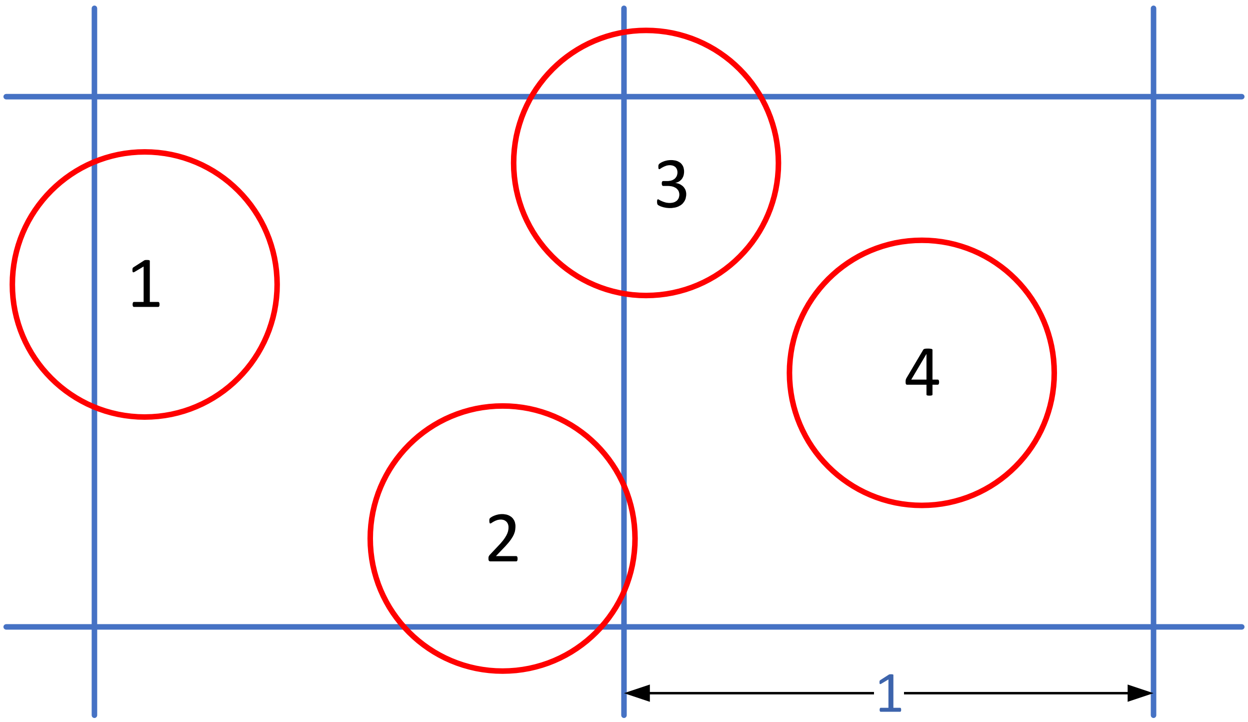

Figure 1

To solve this problem, we begin by noting that knowing a disk's or circle's center coordinates and radius tells us all we need to know about its disposition on the grid. A little thought (and perhaps doodling with pencil and paper) reveals that the approach to determining the allowable scoring area in each square requires different graphical approaches depending on whether the circle's radius R is bigger or smaller than 0.5. For $0 \leq R \leq 0.5$ or R in [0, 0.5], we cannot let the circle get closer than R to any corner, and also not further away than R from any edge of a square. This leaves the light blue scoring area shown in the first quadrant of Figure 2. For R in the interval (0.5, R_{max}], the allowable scoring area is shown in the third quadrant of Figure 2. Due to the symmetries involved, solving for the scoring area in one quadrant solves them for all quadrants of the unit square.

From the first quadrant we compute the scoring area A_S for a given radius R by subtracting the areas of the four corner circle sectors and the center square with sides 1-2R from the unit area of the unit square.

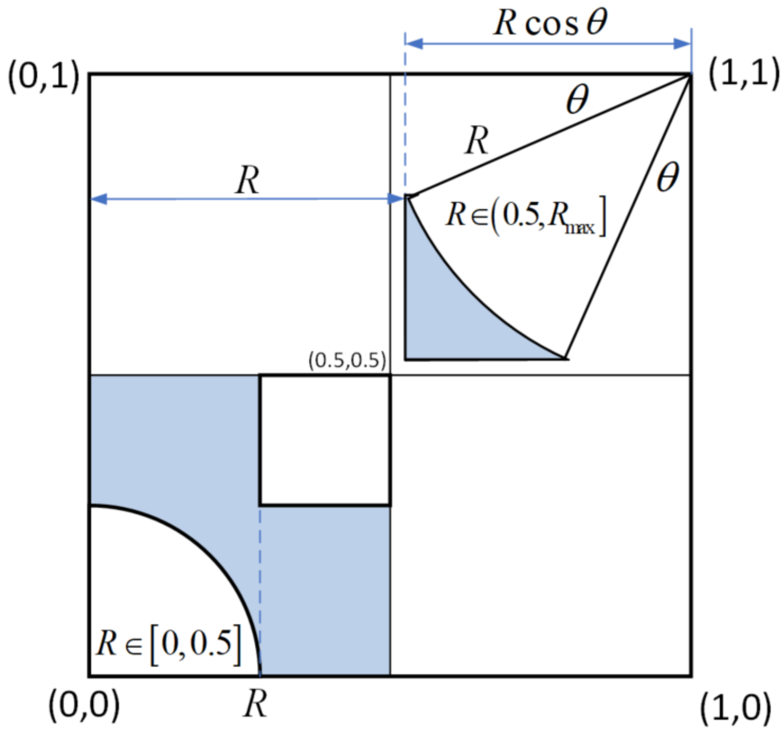

Figure 2

$$
\begin{aligned}
A_S &= 1-\pi R^2-(1-2R)^2 \\
&= 4R-(\pi+4)R^2
\end{aligned}
$$

A_S divided by the area of the unit square (=1) yields the probability of scoring $P(R)$ for $R\in[0,0.5]$. Therefore $P(R) = A_S / 1$ or $P(R)=4R-(\pi+4)R^2$. The optimum radius R_{opt} that maximizes $P(R)$ is found in the usual way through the calculus, and solving for R. This gives

$$
\frac{dP}{dR}=4-2(\pi+4)R=0\rightarrow R_{opt}=\frac{2}{\pi+4}\simeq 0.28
$$

$$
P(R_{opt})=\frac{4}{\pi+4}\simeq 0.56
$$

In the domain $R\in(0.5,R_{max}]$ we see from the third quadrant of Figure 2 that the center square has vanished and the sector of radius R centered at the square's corners has begun to shrink as shown. This is because the growing radius is now constrained to satisfy $R+R\cos\theta=1$, which defines the curved arc of the kite-shaped scoring area. Note that we can roll the circle along the indicated straight line boundaries and still satisfy the scoring requirement. Finally, R_{max} is obtained when the arc vanishes to a point as $\theta\rightarrow \pi/4$. At $\theta=\pi/4$ we solve for R_{max} and the single point at which such a circle can still satisfy the scoring requirements. This yields

$$R_{\max} + R_{\max}\cos\left(\pi/4\right) = 1 \to R_{\max} = \frac{1}{1+\cos\left(\pi/4\right)} \simeq 0.5858$$

And to finish the $P(R)$ calculation for the $R \in \left(0.5, R_{\max}\right]$ domain, we compute the kite-shaped scoring area by subtracting out all the parts of the reduced (1-R) sided square from which scoring is not possible (recalling the symmetry argument for the quadrants).

$$\begin{aligned} A_S &= (1-R)^2 - (R\sin\theta)(R\cos\theta) - \left(\frac{\pi/2 - 2\theta}{2\pi}\right)\pi R^2 \\ &= (1-R)^2 - \left[\frac{\sin 2\theta}{2} + \left(\frac{\pi}{4} - \theta\right)\right]R^2 \\ P(R) = 4A_S &= 4(1-R)^2 - \left[2\sin 2\theta + (\pi - 4\theta)\right]R^2 \end{aligned}$$

where $\theta(R) = \cos^{-1}\left(\frac{1-R}{R}\right)$.

When we plot $P(R)$ over its full range [0, R_{max}], the two solution regimes are apparent in the right-hand tail where the slope changes at R = 0.5. To verify the above calculations, the problem was cast as a Monte Carlo simulation at R = [0:0.01: R_{max}] with 10K samples at each radius value. The two plots (theoretical = green, simulation = red) are shown in Figure 3.

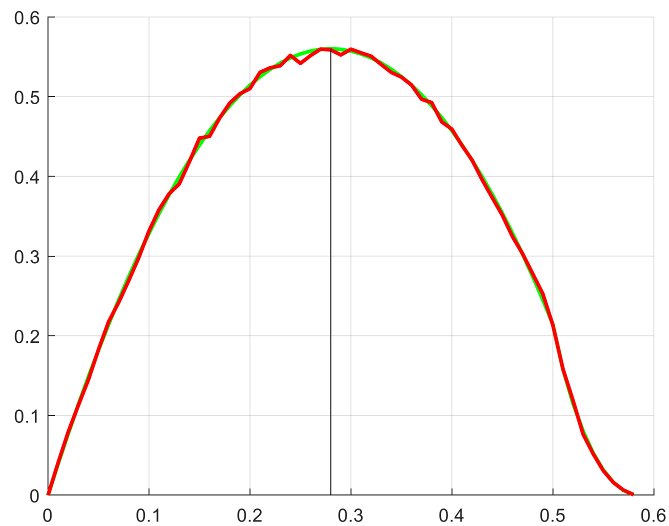

Figure 3

