TN2004-3: Bootstrapping Parameters for a New Test

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In TN2004-2[1] we developed the formula for computing the estimated population fraction from the count of tests positive that obtained from administering the test to a random sample from the target population. There the test's performance parameters – sensitivity $P(TP|V)$,and specificity $P(\neg TP|\neg V)$ – were known. In these probabilities TP is the test positive event, and V is the given presence of the virus. The usual 'not' symbols are understood. The correction formula is

$$f_{IA} = \frac{f_{TP} - P(TP|\neg V)}{P(TP|V) - P(TP|\neg V)} = \frac{f_{TP} - P^-}{P^+ - P^-} \ , \qquad (1)$$

Where the fractions f_{IA} and f_{TP} denote the estimates of the actual infected fraction and the fraction of tests positive from the sample respectively. For brevity we will use $P^+ = P(TP|V)$, and $P^- = P(TP|\neg V) = 1 - P(\neg TP|\neg V)$ in the sequel.

The problem addressed in this technical note is how to determine the performance parameters of a new (N) test which is more desirable than the established old (O) test for reasons having, perhaps, to do with cost, ease of application, rapid results turnaround, etc. We want to get the new test online as quickly as possible, which requires that we also can quickly determine its performance parameters without having to go through the more lengthy and costly formal process.

We proceed as follows by first obtaining two actual infected fractions generated from two target populations with the old test – label them f_{OA1} and f_{OA2}. We test the same target populations with the new test and obtain a tests positive fraction for each, which we label f_{NT1} and f_{NT2}. These are related to the correct actual fractions through (1), giving us two equations for the desired two unknowns P_N^+ and P_N^-.

$$f_{OA1} = \frac{f_{NT1} - P_N^-}{P_N^+ - P_N^-}, \quad f_{OA2} = \frac{f_{NT2} - P_N^-}{P_N^+ - P_N^-} \qquad (2)$$

These are easily solved for the desired new test performance parameters, and yield

$$P_N^+ = \frac{f_{NT1}(1 - f_{OA2}) - f_{NT2}(1 - f_{OA1})}{f_{OA1} - f_{OA2}}, \quad P_N^- = \frac{f_{NT2} f_{OA1} - f_{NT1} f_{OA2}}{f_{OA1} - f_{OA2}} \qquad (3)$$

Since the variances for f_{OA1} and f_{OA2} are known, the variances for these newly estimated parameters may be developed in a manner similar to the one shown in the referenced TN2004-2. We can then write the relations for the desired variances as –

[1] Rebane, G.J., *TN2004-2:Testing for Population Fraction*, 4 April 2020

$$
\begin{aligned}
\sigma_{P^+}^2 &= \left(\frac{\partial f_{P^+}}{\partial f_{NT1}}\right)^2 \sigma_{f_{NT1}}^2 + \left(\frac{\partial f_{P^+}}{\partial f_{NT2}}\right)^2 \sigma_{f_{NT2}}^2 + \left(\frac{\partial f_{P^+}}{\partial f_{OA1}}\right)^2 \sigma_{f_{OA1}}^2 + \left(\frac{\partial f_{P^+}}{\partial f_{OA2}}\right)^2 \sigma_{f_{OA2}}^2 \\
\sigma_{P^-}^2 &= \left(\frac{\partial f_{P^-}}{\partial f_{NT1}}\right)^2 \sigma_{f_{NT1}}^2 + \left(\frac{\partial f_{P^-}}{\partial f_{NT2}}\right)^2 \sigma_{f_{NT2}}^2 + \left(\frac{\partial f_{P^-}}{\partial f_{OA1}}\right)^2 \sigma_{f_{OA1}}^2 + \left(\frac{\partial f_{P^-}}{\partial f_{OA2}}\right)^2 \sigma_{f_{OA2}}^2
\end{aligned}
, \qquad (4)
$$

where the partial derivatives given below are evaluated with the supplied corrected estimates and the fractions sampled with the new test. The partial derivatives are as follows.

$$
\begin{aligned}
&\frac{\partial f_{P^+}}{\partial f_{NT1}} = \frac{1 - f_{OA2}}{f_{OA1} - f_{OA2}}, \quad \frac{\partial f_{P^+}}{\partial f_{NT2}} = \frac{-\left(1 - f_{OA1}\right)}{f_{OA1} - f_{OA2}}, \\
&\frac{\partial f_{P^+}}{\partial f_{OA1}} = \frac{\left(1 - f_{OA2}\right)\left(f_{NT2} - f_{NT1}\right)}{\left(f_{OA1} - f_{OA2}\right)^2}, \quad \frac{\partial f_{P^+}}{\partial f_{OA2}} = \frac{\left(1 - f_{OA1}\right)\left(f_{NT1} - f_{NT2}\right)}{\left(f_{OA1} - f_{OA2}\right)^2}, \\
&\frac{\partial f_{P^-}}{\partial f_{NT1}} = \frac{-f_{OA2}}{f_{OA1} - f_{OA2}}, \quad \frac{\partial f_{P^-}}{\partial f_{NT2}} = \frac{f_{OA1}}{f_{OA1} - f_{OA2}}, \\
&\frac{\partial f_{P^-}}{\partial f_{OA1}} = \frac{f_{OA2}\left(f_{NT1} - f_{NT2}\right)}{\left(f_{OA1} - f_{OA2}\right)^2}, \quad \frac{\partial f_{P^-}}{\partial f_{OA2}} = \frac{f_{OA1}\left(f_{NT2} - f_{NT1}\right)}{\left(f_{OA1} - f_{OA2}\right)^2}.
\end{aligned}
\qquad (5)
$$